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RECURRENCE IN UNIPOTENT GROUPS AND ERGODIC NONABELIAN GROUP EXTENSIONS

BY

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ABSTRACT

Let T be a measure-preserving and ergodic transformation of a standard probability space $(X, \mathcal{S}, \mu)$ and let $f\colon X \longrightarrow \mathrm{SUT}_d(\mathbb{R})$ be a Borel map into the group of unipotent upper triangular $d \times d$ matrices. We modify an argument in [12] to obtain a sufficient condition for the recurrence of the random walk defined by f, in terms of the asymptotic behaviour of the distributions of the suitably scaled maps

$$f(n,x) = (fT^{n-1} \cdot fT^{n-2} \cdots fT \cdot f).$$

We give examples of recurrent cocycles with values in the continuous Heisenberg group $\mathrm{H}_1(\mathbb{R}) = \mathrm{SUT}_3(\mathbb{R})$, and we use a recurrent cocycle to construct an ergodic skew-product extension of an irrational rotation by the discrete Heisenberg group $\mathrm{H}_1(\mathbb{Z}) = \mathrm{SUT}_3(\mathbb{Z})$.

1. Introduction

Let G be a locally compact group, T a measure-preserving and ergodic transformation of a standard probability space $(X, \mathcal{S}, \mu)$ and $f\colon X \longrightarrow G$ a Borel map. We define a map $f\colon \mathbb{Z} \times X \longrightarrow G$ by

$$(1.1) \qquad f(n,x) = \begin{cases} f(T^{n-1}x) \cdots f(Tx) \cdot f(x) & \text{if } n \geq 1, \\ \mathbf{1} & \text{if } n = 0, \\ f(-n, T^n x)^{-1} & \text{if } n < 0. \end{cases}$$

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This map satisfies that

$$f(k, T^l x) \cdot f(l, x) = f(k + l, x) \tag{1.2}$$

for every $k, l \in \mathbb{Z}$, $x \in X$, and is thus a cocycle of the action $(m, x) \longrightarrow T^m x$ of $\mathbb{Z}$ on X.

Definition 1.1: The function $f: X \longrightarrow G$ is **recurrent** if, for every $B \in \mathcal{S}$ with $\mu(B) > 0$, and for every neighbourhood $\mathcal{U}$ of the identity in G,

$$\mu(B \cap T^{-n} B \cap \{x: f(n, x) \in \mathcal{U}\}) > 0 \tag{1.3}$$

for some $n \neq 0$. A function which is not recurrent is **transient**. The recurrence of a function also has an interpretation in terms of the **skew product** transformation

$$\mathbf{T}_f(x, g) = (Tx, f(x) \cdot g) \quad \text{on } (X \times G, \mathcal{S} \times \mathcal{B}_G, \mu \times \lambda_G),$$

in the sense that f is recurrent if and only if $\mathbf{T}_f$ is **conservative**, i.e. for any subset $B \in \mathcal{S} \times \mathcal{B}_G$ of positive measure we have

$$(\mu \times \lambda_G)\Big(\bigcup_{n \in \mathbb{Z}} B \cap \mathbf{T}_f^{-n} B \cap \{(x, g): \mathbf{T}_f^n(x, g) \neq (x, g)\} \Big) > 0.$$

Definition 1.2: A group element $g \in G$ is called an **essential value** of a function $f: X \longrightarrow G$ if, for every $B \in \mathcal{S}$ with $\mu(B) > 0$ and for every neighbourhood $\mathcal{U}(g)$ of g,

$$\mu(B \cap T^{-n} B \cap \{x: f(n, x) \in \mathcal{U}(g)\}) > 0 \tag{1.4}$$

for some $n \neq 0$. Hence a function is recurrent if and only if the identity element is an essential value. The set of all essential values of f is denoted by $E(f)$.

In the case $G = \mathbb{R}$ in [11] a sufficient condition for recurrence is developed and in [12] this condition is generalised to $G = \mathbb{R}^d$. Here we deal with an extension of these results to a nonabelian group $G = \mathrm{SUT}_d(\mathbb{R})$ of unipotent upper triangular matrices.

For the statement of the recurrence condition we have to consider the distributions of the suitably scaled functions $f(n, x)$. In the case $G = \mathbb{R}^d$ (cf. [12]) the distributions of the functions $f(n, x)/n^{1/d}$ are considered but for $G = \mathrm{SUT}_d(\mathbb{R})$ we have to introduce a one-parameter group $\{\alpha_t: t \in \mathbb{R}^\times\}$ of automorphisms to scale the functions; for any matrix

$$\mathbf{A} = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & \cdots & a_{1d} \\ 0 & 1 & a_{23} & a_{24} & \cdots & a_{2d} \\ 0 & 0 & 1 & a_{34} & \cdots & a_{3d} \\ 0 & 0 & 0 & 1 & \cdots & a_{4d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathrm{SUT}_d(\mathbb{R})$$

we set

$$\alpha_t(\mathbf{A}) = \begin{pmatrix} 1 & ta_{12} & t^2a_{13} & t^3a_{14} & \cdots & t^{d-1}a_{1d} \\ 0 & 1 & ta_{23} & t^2a_{24} & \cdots & t^{d-2}a_{2d} \\ 0 & 0 & 1 & ta_{34} & \cdots & t^{d-3}a_{3d} \\ 0 & 0 & 0 & 1 & \cdots & t^{d-4}a_{4d} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{1.5}$$

Furthermore, we need a right-invariant metric d on $\mathrm{SUT}_d(\mathbb{R})$ which is compatible with the automorphisms $\{\alpha_t \colon t \in \mathbb{R}^\times\}$. The norm N on $\mathrm{SUT}_d(\mathbb{R})$ defined by

$$N(\mathbf{A}) = \max_{1\le i\le d-1}\max_{1\le j\le d-i}\max\{|(\mathbf{A})_{j,j+i}|, |(\mathbf{A}^{-1})_{j,j+i}|\}^{1/i} \tag{1.6}$$

fulfils that

$$N(\mathbf{A}) = N(\mathbf{A}^{-1}), N(\mathbf{AB}) \le N(\mathbf{A}) + N(\mathbf{B}).$$

Indeed, the first property follows immediate from the definition, which is symmetric in $\mathbf{A}$ and $\mathbf{A}^{-1}$. To verify the sub-linearity we fix the integers $1 \le i \le d-1$ and $1 \le j \le d-i$ and use the binomial law to conclude that

$$\begin{aligned} |(\mathbf{AB})_{j,j+i}| &= \left|\sum_{k=1}^{d}(\mathbf{A})_{j,k}\cdot(\mathbf{B})_{k,j+i}\right| = \left|\sum_{k=j}^{j+i}(\mathbf{A})_{j,k}\cdot(\mathbf{B})_{k,j+i}\right| \\ &\le \sum_{k=j}^{j+i}|(\mathbf{A})_{j,k}|\cdot|(\mathbf{B})_{k,j+i}| \le \sum_{k=j}^{j+i}N(\mathbf{A})^{k-j}\cdot N(\mathbf{B})^{i-k+j} \\ &\le \sum_{k=j}^{j+i}\binom{i}{k-j}\cdot N(\mathbf{A})^{k-j}\cdot N(\mathbf{B})^{i-k+j} = (N(\mathbf{A})+N(\mathbf{B}))^i. \end{aligned}$$

Furthermore, this norm is compatible with the automorphisms $\{\alpha_t \colon t \in \mathbb{R}^\times\}$ in the sense that

$$N(\alpha_t(\mathbf{A})) = t\cdot N(\mathbf{A}) \quad \text{for all } 0 < t \in \mathbb{R}.$$

Hence this norm defines a right-invariant metric d on $\mathrm{SUT}_d(\mathbb{R})$ by

$$d(\mathbf{A},\mathbf{B}) = N(\mathbf{A}\mathbf{B}^{-1}), \tag{1.7}$$

and this metric fulfils that

$$d(\alpha_t(\mathbf{A}),\mathbf{1}) = t \cdot d(\mathbf{A},\mathbf{1}) \tag{1.8}$$

for all $0 < t \in \mathbb{R}$.

The group $\mathrm{SUT}_d(\mathbb{R})$ is unimodular for every $d \geq 3$ and the Haar measure is the Lebesgue measure λ on $\mathbb{R}^{d(d-1)/2}$, because $d(d-1)/2$ is the number of coordinates above the main diagonal of a $d \times d$ matrix. The automorphism α_t is bijective for every $t > 0$ and hence $\lambda \circ \alpha_t$ is invariant and therefore also a Haar measure. We conclude that

$$\lambda \circ \alpha_t = t^s \cdot \lambda \quad \text{with } s = (d+1)d(d-1)/6 \tag{1.9}$$

by uniqueness of the Haar measure, where the constant factor is determined to be equal to t^s by the transformation behaviour of $d(d-1)/2$-dimensional unit cube under α_t.

2. Recurrence and transience in unipotent groups

We set $s = (d+1)d(d-1)/6$ and define for every $k \geq 1$ the probability measures $\sigma_k^{(d)}$ and $\tau_k^{(d)}$ of a Borel set $A \in \mathcal{B}(\mathrm{SUT}_d(\mathbb{R}))$ by

$$\sigma_k^{(d)}(A) = \mu(\{x \in X \colon \alpha_{k^{-1/s}}(f(k,x)) \in A\}), \tag{2.1}$$

$$\tau_k^{(d)}(A) = \frac{1}{k}\sum_{l=1}^{k} \sigma_k^{(d)}(A). \tag{2.2}$$

With these requisites we are ready to formulate the main result:

THEOREM 2.1: *Let T be a measure-preserving and ergodic transformation of a standard probability space $(X, \mathcal{S}, \mu)$ and let $f\colon X \longrightarrow \mathrm{SUT}_d(\mathbb{R})$ be a Borel map into the group of unipotent upper triangular $d \times d$ matrices for some $d \geq 3$. Let $B(\eta) = \{\mathbf{A} \in \mathrm{SUT}_d(\mathbb{R})\colon d(\mathbf{A},\mathbf{1}) < \eta\}$ for every $\eta > 0$ and let λ denote the Haar measure on $\mathrm{SUT}_d(\mathbb{R})$. If f is transient, then we have the equations*

$$\sup_{\eta>0} \limsup_{k\to\infty} \tau_k^{(d)}(B(\eta))/\lambda(B(\eta)) < \infty \tag{2.3}$$

and

$$\liminf_{\eta\to 0} \liminf_{k\to\infty} \tau_k^{(d)}(B(\eta))/\lambda(B(\eta)) = 0. \tag{2.4}$$

These equations are formally the same as in [12], Theorem 1.2.

Remark 2.2: The theorem also applies to any connected closed subgroup G of $\mathrm{SUT}_d(\mathbb{R})$ which is invariant under $\{\alpha_t\colon t \in \mathbb{R}^\times\}$, but the value of s has to be reduced to fulfil equation (1.9), where λ is the Haar measure on G. The relation between $\lambda \circ \alpha_t$ and λ is always a power of t ([5], Chapter 1, Section 2, Lemma 1). In the simplest case of a closed connected proper subgroup G there are blocks above the main diagonal equal to zero, while all the remaining components are independent, and under these conditions we have that

$$s = \sum_{i=1}^{d-1} i \cdot \#\{1 \le k \le d-i\colon a_{k,k+i} \ne 0 \text{ for some } \mathbf{A} \in G\}$$

while λ is the Lebesgue measure on $\mathbb{R}^l$ with

$$l = \sum_{i=1}^{d-1} \#\{1 \le k \le d-i\colon a_{k,k+i} \ne 0 \text{ for some } \mathbf{A} \in G\}.$$

The proof of Theorem 2.1 depends on the following lemma:

LEMMA 2.3: *Suppose that $f\colon X \longrightarrow \mathrm{SUT}_d(\mathbb{R})$ is transient, and let $C \subset X$ be a Borel set with $\mu(C) > 0$ and $\varepsilon > 0$ a real number with*

$$\mu(C \cap T^{-n}C \cap \{x \in X\colon d(f(n,x),\mathbf{1}) < \varepsilon\}) = 0 \tag{2.5}$$

for all $n \ne 0$. By replacing C by a subset, if necessary, we can assume without loss of generality that $\mu(C) = 1/L$ for some integer L. Then we have, for any $\eta > 0$ and $N \ge 1$, the inequalities

$$\limsup_{k\to\infty} \tau_k^{(d)}(B(\eta)) \le 2^s L \cdot \lambda(B(\eta))/\lambda(B(\varepsilon)) \tag{2.6}$$

and

$$\limsup_{k\to\infty} \sum_{n=0}^{N} 2^n \tau_{2^n+k}^{(d)}(B(2^{-n/s}\eta)) \le 5^{s-1}8Ls^2 \cdot \lambda(B(\eta))/\lambda(B(\varepsilon)). \tag{2.7}$$

Proof of Theorem 2.1 given Lemma 2.3: The inequality (2.3) is an immediate implication of the inequality (2.6). For the proof of the equation (2.4) we fix some $\eta > 0$ and conclude from equation (2.7) that

$$\limsup_{k\to\infty} \sum_{n=0}^{N} \frac{\tau_{2^n+k}^{(d)}(B(2^{-n/s}\eta))}{\lambda(B(2^{-n/s}\eta))} \le c = 5^{s-1}8Ls^2/\lambda(B(\varepsilon)),$$

and hence

$$\liminf_{k\to\infty} \frac{\tau^{(d)}_{2^{n+k}}(B(2^{-n/s}\eta))}{\lambda(B(2^{-n/s}\eta))} \leq c/(N+1)$$

for some $0 \leq n \leq N$. As N was arbitrary this implies the hypothesis. ∎

Proof of Lemma 2.3: The proof uses the same methods as the proof of Lemma 2.1 in [12], but now these methods are applied to a nonabelian group and we use that the metric d is compatible with the one-parameter group $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms. First we modify T on a null-set, if necessary, to eliminate any periodic points. Let

$$R_T = \{(T^n x, x) : x \in X, n \in \mathbb{Z}\} \subset X \times X \tag{2.8}$$

denote the **orbit equivalence relation** of T, and define a map $\mathbf{f}$: $R_T \longrightarrow \mathrm{SUT}_d(\mathbb{R})$ by

$$\mathbf{f}(T^n x, x) = f(n, x) \quad \text{for every } x \in X, n \in \mathbb{Z}. \tag{2.9}$$

The map $\mathbf{f}$ fulfils the equation

$$\mathbf{f}(x, x') \cdot \mathbf{f}(x', x'') = \mathbf{f}(x, x'') \tag{2.10}$$

for every $(x, x'), (x', x'') \in R_T$, and hence is a **cocycle** of the orbit equivalence relation R_T.

The *full group* $[T]$ of T consists of all measure preserving Borel automorphisms W of $(X, \mathcal{S}, \mu)$ with $(Wx, x) \in R_T$ for all $x \in X$, and the ergodicity of T implies that we can find, for any two sets $B_1, B_2 \in \mathcal{S}$ with $\mu(B_1) = \mu(B_2)$, an automorphism $W \in [T]$ with $\mu(WB_1 \Delta B_2) = 0$. Now choose a partition $\{C_i\}_{i=0}^{L-1} \subset \mathcal{S}$ with $C_0 = C, \mu(C_i) = 1/L$ and an automorphism $W \in [T]$ with $\mu(WC_i \Delta C_{i+1}) = 0$ for $i = 0, \ldots, L-2$ and $W^L = \mathrm{Id}_X$. We put

$$Sx = \begin{cases} Wx & \text{if } x \in \bigcup_{k=0}^{L-2} C_k, \\ T_C W x & \text{if } x \in C_{L-1}, \end{cases} \tag{2.11}$$

where $T_C x = T^{m_C(x)} x$ is the transformation induced by T on C, with $m_C(x)$ being the first recovery time of x to C. Then there exists a T-invariant null-set $N \in \mathcal{S}$ so that the following conditions hold:

(1) the sets $C'_k = S^k C'$ are disjoint for $k = 0, \ldots, L-1$, where $C' = C \setminus N$, and $S^k C' = C'$,
(2) $N = X \setminus \bigcup_{k=0}^{L-1} C'_k$,

(3) the sets $\{j \geq 0 \colon S^j x \in C'\}$ and $\{j \leq 0 \colon S^j x \in C'\}$ are both infinite for all $x \in C'$.

Furthermore, the automorphisms T and S are **orbit-equivalent** for all $x \in C'$, i.e. $\{S^n x \colon n \in \mathbb{Z}\} = \{T^n x \colon n \in \mathbb{Z}\}$ holds for a.e. $x \in X$.

We define a Borel map $b \colon X \longrightarrow \mathrm{SUT}_d(\mathbb{R})$ by

$$b(x) = \begin{cases} \mathbf{1} & \text{if } x \in N \\ \mathbf{f}(S^L y, x) & \text{if } x = S^k y \text{ with } y \in C', 1 \leq k \leq L \end{cases}$$

and obtain that the map $g(x) = b(Sx) \cdot \mathbf{f}(Sx, x) \cdot b(x)^{-1}$ satisfies that

$$g(x) = \begin{cases} \mathbf{f}(S^L x, x) = f(m_{C'}(x), x) & \text{if } x \in C', \\ \mathbf{1} & \text{otherwise.} \end{cases} \tag{2.12}$$

If b is used as a **transfer function** for f, then the functions $f'(n,x)$ and $\mathbf{f}'(x,x')$ defined with $f'(x) = b(Tx) \cdot f(x) \cdot b(x)^{-1}$ replacing f fulfil

$$\begin{aligned} f'(n,x) &= b(T^n x) \cdot f(n,x) \cdot b(x)^{-1}, \\ \mathbf{f}'(x,x') &= b(x) \cdot \mathbf{f}(x,x') \cdot b(x'), \end{aligned} \tag{2.13}$$

for every $x \in X \setminus N$ and $x' \in \{S^n x \colon n \in \mathbb{Z}\} = \{T^n x \colon n \in \mathbb{Z}\}$.

The **skew product** transformation $\mathbf{S}_{f'}$ on $(Y, \nu) = (X \times \mathrm{SUT}_d(\mathbb{R}), \mu \times \lambda)$ is defined by

$$\mathbf{S}_{f'}(x, \mathbf{A}) = (Sx, f'(Sx, x) \cdot \mathbf{A}) = (Sx, g(x) \cdot \mathbf{A}) \tag{2.14}$$

and we conclude that the set $D = C' \times B(\varepsilon/2)$ is **wandering** under $\mathbf{S}_{f'}$, i.e. that $\mathbf{S}_{f'}^m D \cap D = \emptyset$ for all $0 \neq m \in \mathbb{Z}$, because equations (2.5) and (2.13) imply that

$$C' \cap S^{-1} C' \cap \{x \in X \colon Vx \neq x \text{ and } d(\mathbf{f}'(Sx,x), \mathbf{1}) < \varepsilon\} = \emptyset. \tag{2.15}$$

We let for every $x \in X \setminus N$

$$V_x = \{f'(k,x) \colon k \in \mathbb{Z}\} = \{\mathbf{f}'(S^k x, x) \colon k \in \mathbb{Z}\} \subset \mathrm{SUT}_d(\mathbb{R}),$$

and observe that for every $\mathbf{A} \in V_x$

$$|\{k \in \mathbb{Z} \colon f'(k,x) = \mathbf{A}\}| = |\{k \in \mathbb{Z} \colon \mathbf{f}'(S^k x, x) = \mathbf{A}\}| = L. \tag{2.16}$$

It follows that

$$\begin{aligned} &|\{0 < l \leq k \colon 0 < d(\alpha_{l^{-1/s}}(f'(l,x)), \mathbf{1}) \leq \eta\}| \\ &= |\{0 < l \leq k \colon 0 < d(f'(l,x), \mathbf{1}) \leq l^{1/s}\eta\}| \\ &\leq |\{0 < l \leq k \colon 0 < d(f'(l,x), \mathbf{1}) \leq k^{1/s}\eta\}| \\ &< \nu(X \times B(k^{1/s}\eta + \varepsilon/2))/\nu(D), \end{aligned}$$

since $\mathbf{S}^l_{f'}D \subset X \times B(k^{1/s}\eta + \varepsilon/2)$. Integration gives

$$\begin{aligned}\tau'_k(B(\eta)) &= \frac{1}{k}\sum_{l=1}^{k}\sigma'_l(B(\eta))\\ &= \frac{1}{k}\sum_{l=1}^{k}\mu(\{x \in X\colon d(\alpha_{l^{-1/s}}(f'(l,x)),\mathbf{1}) \le \eta\}\\ &< \frac{L}{k} + \frac{\nu(X \times B(k^{1/s}\eta+\varepsilon/2))}{k\nu(D)} \le \frac{L}{k} + \frac{L(k^{1/s}\eta+\varepsilon/2)^s}{k(\varepsilon/2)^s},\end{aligned}$$

where σ'_l and τ'_k are defined by the equations (2.1)–(2.2) with f' replacing f. We conclude that

$$\limsup_{k\to\infty}\tau'_k(B(\eta)) < 2^s L \cdot \lambda(B(\eta))/\lambda(B(\varepsilon)). \tag{2.17}$$

Another argument with the wandering set D shows with equation (1.9) that

$$\begin{aligned}|\{\mathbf{A} \in V_x \cap (B((j+1)\varepsilon) \setminus B(j\varepsilon))\}| &\le \frac{\lambda(B((j+3/2)\varepsilon) \setminus B((j-1/2)\varepsilon))}{\lambda(B(\varepsilon/2))}\\ &\le 2^s((j+3/2)^s - (j-1/2)^s) < 2^{s+1}s(j+3/2)^{s-1}\end{aligned} \tag{2.18}$$

for every $j \ge 1$. Furthermore, for every integer $l > 0$ and every real number $\beta > 0$ we observe that

$$\begin{aligned}|\{n \ge 0\colon l \le 2^n \le l\beta\}| &= |\{n \ge 0\colon 0 \le n - \log_2 l \le \log_2\beta\}|\\ &\le \max\{0, 1 + [\log_2\beta]\},\end{aligned}$$

in which $[x] = \max\{i \in \mathbb{Z}\colon i \le x\}$ for any real number x. Then we use this inequality and the equations (2.16), (2.18) in the following computation:

$$\begin{aligned}&\sum_{n\ge 0}|\{0 < l \le 2^n\colon 0 < d(\alpha_{l^{-1/s}}f'(l,x),\mathbf{1}) \le 2^{-n/s}\eta\}|\\ &= \sum_{l>0}|\{n \ge 0\colon 2^n \ge l \text{ and } 0 < 2^n d(f'(l,x),\mathbf{1})^s \le l\eta^s\}|\\ &= \sum_{\mathbf{1}\ne\mathbf{A}\in V_x}\ \sum_{\{l>0\colon f'(l,x)=\mathbf{A}\}}|\{n \ge 0\colon l \le 2^n \le l\eta^s/d(\mathbf{A},\mathbf{1})^s\}|\\ &\le L\sum_{\mathbf{1}\ne\mathbf{A}\in V_x}\max\{0, 1 + [\log_2(\eta^s/d(\mathbf{A},\mathbf{1})^s)]\}\\ &\le L\sum_{j\ge 1}\ \sum_{\mathbf{A}\in V_x\cap(B((j+1)\varepsilon)\setminus B(j\varepsilon))}\max\{0, 1 + [\log_2(\eta^s/j^s\varepsilon^s)]\}\end{aligned}$$

$$
\begin{aligned}
&< 2^{s+1}Ls\sum_{j\geq 1}(j+3/2)^{s-1}\max\{0,1+[\log_2(\eta^s/j^s\varepsilon^s)]\}\\
&\leq 2^{s+1}Ls\sum_{j=1}^{[\eta/\varepsilon]}(j+3/2)^{s-1}(1+\log_2(\eta^s/j^s\varepsilon^s)))\\
&< 2^{s+2}Ls^2\sum_{j=1}^{[\eta/\varepsilon]}(j+3/2)^{s-1}(1+\ln(\eta/j\varepsilon))\\
&\leq 2^{s+2}Ls^2\sum_{j=1}^{[\eta/\varepsilon]}(j+3/2)^{s-1}\frac{\eta}{j\varepsilon} < 2^{s+2}\left(\frac{5}{2}\right)^{s-1}Ls^2\frac{\eta^s}{\varepsilon^s} = 5^{s-1}8Ls^2\frac{\eta^s}{\varepsilon^s}.
\end{aligned}
$$

By integration it follows that

$$
\sum_{n=0}^{\infty}2^n\tau'_{2^n}(B(2^{-n/s}\eta)\setminus\{0\}) < 5^{s-1}8Ls^2\frac{\eta^s}{\varepsilon^s},
$$

and substituting η by $2^{k/s}\eta$, omitting the first k terms in the series, and dividing both sides of the inequality by 2^k shows that

$$
\sum_{n=0}^{\infty}2^n\tau'_{2^{n+k}}(B(2^{-n/s}\eta)\setminus\{0\}) < 5^{s-1}8Ls^2\frac{\eta^s}{\varepsilon^s}
$$

holds for every $k\geq 0$. So we can conclude for every $k\geq 0$ that

$$
\begin{aligned}
\sum_{n=0}^{N}2^n\tau'_{2^{n+k}}(B(2^{-n/s}\eta)) &\leq \sum_{n=0}^{N}\Big(\frac{L}{2^k}+2^n\tau'_{2^{n+k}}(B(2^{-n/s}\eta)\setminus\{0\})\Big)\\
&< \frac{(N+1)L}{2^k}+5^{s-1}8Ls^2\frac{\eta^s}{\varepsilon^s},
\end{aligned}
$$

and in the limit we obtain the inequality

$$
\limsup_{k\to\infty}\sum_{n=0}^{N}2^n\tau'_{2^{n+k}}(B(2^{-n/s}\eta)) < 5^{s-1}8Ls^2\cdot\lambda(B(\eta))/\lambda(B(\varepsilon)). \tag{2.19}
$$

The equations (2.13) imply that

$$
\begin{aligned}
&\liminf_{k\to\infty}(\sigma_k^{(d)}(B(\eta+\eta'))-\sigma'_k(B(\eta)))\geq 0,\\
&\liminf_{k\to\infty}(\sigma'_k(B(\eta+\eta'))-\sigma_k^{(d)}(B(\eta)))\geq 0,
\end{aligned}
$$

for every $\eta,\eta'>0$, because

$$
\begin{aligned}
\alpha_{k^{-1/s}}(f'(k,x)) &= \alpha_{k^{-1/s}}(b(T^kx)\cdot f(k,x)\cdot b(x)^{-1})\\
&= \alpha_{k^{-1/s}}(b(T^kx))\cdot\alpha_{k^{-1/s}}(f(k,x))\cdot\alpha_{k^{-1/s}}(b(x)^{-1})
\end{aligned}
$$

and the sequences $\alpha_{k^{-1/s}}(b(T^k x))$ as well as $\alpha_{k^{-1/s}}(b(x)^{-1})$ converge to the identity in measure as k goes to infinity. Hence equations (2.17) and (2.19) imply the hypotheses of the Lemma. ∎

Example 2.4 (A recurrent cocycle in the Heisenberg group): Let us denote the elements of the Heisenberg group $\mathrm{H}_1(\mathbb{R}) = \mathrm{SUT}_3(\mathbb{R})$ by

$$[a,b,c] = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

for all $a,b,c \in \mathbb{R}$. We define a function F by $F(x) = [f(x), g(x), h(x)]$ and conclude by iteration that

$$F(n,x) = \left[f(n,x), g(n,x), h(n,x) + \sum_{r=0}^{n-2} f(T^{r+1}x) g(r+1,x) \right] \tag{2.20}$$

for every $n \in \mathbb{Z}$.

Let $(X, \mathcal{S}, \mu)$ be the torus $\mathbb{R}/\mathbb{Z}$ with the Borel σ-algebra and the Lebesgue measure, and let $\|x\|$ denote the positive distance from $x \in \mathbb{R}$ to the nearest integer. The automorphism $T = T_\alpha$ is the rotation modulo one by an irrational number $\alpha = [0; a_1, a_2, \ldots]$ with **bounded** continued fraction coefficients $\{a_k\}_{k \geq 1}$ and maximum $a = \max_{k \geq 1}\{a_k\}$ (e.g. any **algebraic** irrational number of order 2). The **convergents** are defined by $p_i/q_i = [0, a_1, \ldots, a_i]$, and the denominators q_i can be computed inductively by $q_{-1} = 0$, $q_0 = 1$, and $q_i = a_i q_{i-1} + q_{i-2}$. An easy induction argument shows that $q_i \geq (3/2)^{i-1}$ and thus the inequality

$$i < (5/2) \log 2q_i$$

holds for all $i \geq 1$.

We choose **piecewise constant** functions f and g with integral zero which obey the relations

$$f(x + 1/2) = f(x) \quad \text{and} \quad g(x+1/2) = -g(x)$$

for all $x \in X$, while h is an arbitrary function of **bounded variation** with integral zero. For any given integer n one can find uniquely determined integers m and $b_i, 0 \leq i \leq m$ with the following properties (see [7], p. 148):

(1) $n = \sum_{i=0}^m b_i q_i$,
(2) $b_m > 0$ and $0 \leq b_i \leq a_{i+1}$.

The inequality of **Denjoy–Koksma** says that

$$\left\| k(q_i, \cdot) - q_i \int k d\mu \right\|_\infty \leq \mathrm{Var}(k)$$

for any q_i, whenever k is a function of bounded variation. So we obtain that

$$\|f(n,\cdot)\|_\infty \leq ma \,\mathrm{Var}(f) < a \,\mathrm{Var}(f)(5/2)\log 2n,$$

and accordingly for the functions g and h.

For the examination of the sum in the third component of the cocycle $F(n,x)$ we rewrite the term as

$$\sum_{\substack{0 \leq r \leq n-2 \\ 0 \leq s \leq r}} f(T^{r+1}x)g(T^s x)$$

and observe that the range of indices (r,s) varies over a triangular set in $\mathbb{Z}^2$. If n is even we consider the partial sum over the last column of the index set separately, and for this column of indices it follows for all $x \in X$ that

$$\left| \sum_{0 \leq s \leq n-2} f(T^{n-1}x)g(T^s x) \right| \leq a\|f\|_\infty \mathrm{Var}(g)(5/2)\log 2n.$$

Now we can assume n to be odd and set $(n-1)/2 = \sum_{i=0}^m b_i q_i$. Then the triangular index set can be decomposed into the following parts, in which $b = \sum_{i=0}^m b_i$:

- $2b(b-1)$ rectangular sets $\{(r,s)\colon t \leq r \leq t+q_i-1, u \leq s \leq u+q_j-1\}$ with $0 \leq u < t < n-1$ and $0 \leq i,j \leq m$;
- b_i triangular sets $\{(r,s)\colon t \leq r \leq t+2q_i-1, t \leq s \leq r\}$ with some $0 \leq t < n-1$ for every $0 \leq i \leq m$, hence altogether b triangular sets.

The partial sum over some rectangular set is equal to

$$\sum_{\substack{t \leq r \leq t+q_i-1 \\ u \leq s \leq u+q_j-1}} f(T^{r+1}x)g(T^s x) = f(q_j, T^{t+1}x)g(q_i, T^u x)$$

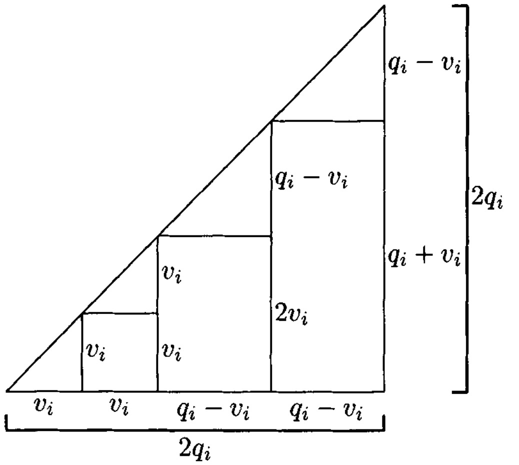
and therefore its modulus is uniformly bounded by $\mathrm{Var}(f)\,\mathrm{Var}(g)$. Hence the partial sum over the $2b(b-1)$ rectangular sets in the partition is uniformly bounded by

$$2b(b-1)\,\mathrm{Var}(f)\,\mathrm{Var}(g) < 2a^2\,\mathrm{Var}(f)\,\mathrm{Var}(g)((5/2)\log 2n)^2.$$

For each $0 \leq i \leq m$ we can choose an integer v_i with $0 < v_i < q_i$ so that both the inequalities $\|v_i\alpha - 1/2\| < 3/2q_i$ and $\|(q_i - v_i)\alpha - 1/2\| < 3/2q_i$ hold, because $|\alpha - p_i/q_i| < 1/q_i^2$, and hence for every $0 \leq l < q_i$

$$\#\{r\colon 0 \leq r < q_i \text{ and } \mathrm{T}^r x \in [l/q_i, (l+1)/q_i)\} = 1. \tag{2.21}$$

Then each triangular set $\{(r,s)\colon t \le r \le t+2q_i-1, t \le s \le r\}$ is decomposed according to the following partition:



The modulus of the partial sum over the three rectangular sets in this partition is uniformly bounded by

$$3a^2 \operatorname{Var}(f) \operatorname{Var}(g)((5/2)\log 2n)^2.$$

The two triangular sets left in the partition are congruent and hence the partial sum over both of them can be written as

$$\sum_{\substack{t \le r \le t+v_i-1 \\ t \le s \le r}} (f(T^{r+1}x)g(T^s x) + f(T^{r+1}T^{v_i}x)g(T^s T^{v_i}x)). \tag{2.22}$$

The choice of v_i implies that $T^{v_i}x = x + \frac{1}{2} + \delta$, where $|\delta| < 3/2q_i$, and thus we have $f(x) = f(T^{v_i}x)$ (respectively $g(x) = -g(T^{v_i}x)$) for all x not in an interval of length $|\delta|$ left from a discontinuity of f (respectively g), if $\delta > 0$, and right from a discontinuity of f (respectively g), if $\delta < 0$. For every $x \in X$ and every interval I of length $|\delta| < 3/2q_i$, equation (2.21) implies that

$$\#\{r\colon 0 \le r < q_i \text{ and } \mathrm{T}^r x \in I\} \le 3.$$

If $D(f)$ denotes the set of discontinuities of f, then there are at most $3 \cdot \#D(f)$ columns and at most $3 \cdot \#D(g)$ rows in the index set of equation (2.22) in which the argument of the sum **can be** different from zero. We sum up one of these columns $\{(r,s)\colon t \le s \le r\}$ with some $r \in \{t,\dots,t+v_i-1\}$ and obtain for all $x \in X$ that

$$\left|\sum_{s=t}^{r}(f(T^{r+1}x)g(T^s x) + f(T^{r+1}T^{v_i}x)g(T^s T^{v_i}x))\right| \le 2a\|f\|_\infty \operatorname{Var}(g)(5/2)\log 2n.$$

The same argument applies for the rows of indices with arguments possibly nonzero, and we observe that the arguments with an index in the intersection of such a column and row are counted twice. With the inequality $\|f\|_\infty \le \operatorname{Var}(f)$ for a function of integral zero it results that equation (2.22) is uniformly bounded by

$$6a(\#D(f)+\#D(g))\operatorname{Var}(f)\operatorname{Var}(g)(5/2)\log 2n + 18\#D(f)\#D(g)\operatorname{Var}(f)\operatorname{Var}(g).$$

The same bound also holds for the two triangular index sets right in the partition scheme, and finally we achieve with some constants $c,d > 0$ and uniformly in $x \in X$ and $n \in \mathbb{N}$ that

$$h(n,x) + \sum_{r=0}^{n-2} f(T^{r+1}x)g(r+1,x) < c(\log 2n)^3 + d.$$

So we can conclude that the sequences $\sigma_k^{(3)}$ as well as $\tau_k^{(3)}$ corresponding to the function F converge to δ_1, the probability measure with mass one on the identity of the Heisenberg group, and due to Theorem 2.1 the function F is recurrent.

Example 2.5 (The simple random walk in the Heisenberg group is transient): Let $(Y,\mathcal{T},\nu)$ be a probability space and $f\colon Y \longrightarrow \mathrm{H}_1(\mathbb{Z})$ a measurable function on Y with a finitely supported symmetric and generating distribution measure, e.g. the simple random walk with

$$\begin{aligned}\nu(\{x\colon f(x)=[0,0,\pm 1]\}) &= \nu(\{x\colon f(x)=[0,\pm 1,0]\})\\ &= \nu(\{x\colon f(x)=[\pm 1,0,0]\}) = 1/6.\end{aligned}$$

Define $(X,\mathcal{S},\mu) = (Y^{\mathbb{Z}},\mathcal{T}^{\mathbb{Z}},\nu^{\mathbb{Z}})$, let T be the shift transformation with $(Tx)_k = x_{k+1}$ for every $x = (\dots,x_{-1},x_0,x_1,\dots)$ and define $F\colon X \longrightarrow \mathrm{H}_1(\mathbb{R})$ by $F(x) = f(x_0)$. Then the cocycle $F(n,\cdot)$ is transient (cf. [13]), because $\mathrm{H}_1(\mathbb{Z})$ does not contain any subgroup of finite index which is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.

3. Ergodic nonabelian group extensions

The last section is devoted to the construction of an **ergodic skew product** extension by the **discrete** Heisenberg group $\mathrm{H}_1(\mathbb{Z}) = \mathrm{SUT}_3(\mathbb{Z})$. The paper [3] shows with abstract methods that any **amenable** locally compact second countable group admits an ergodic skew product extension of an ergodic transformation and then it follows that ergodic skew product extensions are generic (i.e. the set of functions which define an ergodic skew product extension is a dense G_δ-set in the topology of convergence in measure). Our intention here is to give a simple example based on the irrational rotation modulo one on the torus. We already know that a cocycle is recurrent if and only if the identity is an essential value, and the recurrence of the cocycle defined below will be indispensable in the proof of ergodicity. We will make use of the following facts, which are well known from the literature.

PROPOSITION 3.1: *Let $f\colon X \longrightarrow G$ be a Borel function with values in a locally compact second countable group G. The set of essential values $E(f)$, also called the essential range, is a closed subgroup of G. If $H \subset E(f)$ is a closed normal subgroup of G, then $f \cdot H\colon X \longrightarrow G/H$ has essential values $E(f)/H$. The skew product transformation $\mathbf{T}_f$ is ergodic on $(X \times G, \mu \times \lambda_G)$ if and only if $E(f)$ is equal to G. The set of Borel functions $f\colon X \longrightarrow G$ with $E(f) = G$ forms a G_δ-set in $\mathcal{B}(X, G)$ with respect to the topology of convergence in measure.*

Proof: The proofs for abelian G can be found in [10], Theorem 3.9, Corollary 5.4, and Lemmas 9.4–9.8, and the extension to nonabelian G is straightforward. ∎

The question whether the essential range is a cohomology invariant naturally arises, and the affirmative answer is well known for abelian G ([10], Theorem 3.9), while the paper [1] gives examples in $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SO}(3)$ where the essential range of the cohomologous function is not even conjugate to the essential range of the original function. Later, we will give an example that such phenomena also occur in the one-stage nilpotent group $\mathrm{H}_1(\mathbb{Z})$. For nilpotent locally compact second countable groups we have the following result:

PROPOSITION 3.2: *Let G be a nilpotent locally compact second countable group and let $f\colon X \longrightarrow G$ be a Borel function with the closed commutator subgroup $G' = \overline{[G, G]}$ being included in $E(f)$. Then the essential range is a cohomology invariant, i.e. $E(b \circ T \cdot f \cdot b^{-1}) = E(f)$ for every Borel function $b\colon X \longrightarrow G$.*

Proof: The proof is by induction, starting with a one-stage nilpotent (and therefore abelian) group. If G is $n+1$-stage nilpotent then $G/Z(G)$ is n-stage nilpotent and $(G/Z(G))' \subset G'/Z(G) \subset E(f \cdot Z(G))$. With the induction hypotheses it follows that the essential range of $f \cdot Z(G)$ is invariant. If we fix $g \in E(f)$ and choose a neighbourhood $\mathcal{U}(g)$ of g and $B \in \mathcal{S}$ with $\mu(B) > 0$, then for some $m \neq 0$ and suitable $d \in Z(G)$ the set $C = B \cap T^{-m}B \cap \{x\colon b(T^m x)f(m,x)b(x)^{-1}d \in \mathcal{U}(g)\}$ has positive measure. Any element $g' \in Z(G) \subset G'$ is invariant under conjugation and hence $E(f) \cap Z(G) = E(b \circ T \cdot f \cdot b^{-1}) \cap Z(G) = Z(G)$. It follows with a suitably chosen integer $n \neq 0$ that $\mu(C \cap T^{-n}C \cap \{x\colon b(T^n x)f(n,x)b(x)^{-1} = d\}) > 0$ and therefore $\mu(B \cap T^{-m-n}B \cap \{x\colon b(T^{m+n}x)f(m+n,x)b(x)^{-1} \in \mathcal{U}(g)\}) > 0$. As B and $\mathcal{U}(g)$ were arbitrary we conclude that $g \in E(b \circ T \cdot f \cdot b^{-1})$. ∎

Hence the essential range of a function into the Heisenberg group is invariant if the centre, which is equal to the commutator subgroup, is included in the essential range.

Now let $(X, \mathcal{S}, \mu)$ be the torus $\mathbb{R}/\mathbb{Z}$ with the Borel σ-algebra and the Lebesgue measure, while $T = T_\alpha$ is the rotation modulo one by an irrational number α with bounded continued fraction coefficients. Throughout this section we denote by $\|x\|$ the positive distance from $x \in \mathbb{R}$ to the nearest integer. We consider the skew product extension $\mathbf{T}_F\colon X \times \mathrm{H}_1(\mathbb{Z}) \longrightarrow X \times \mathrm{H}_1(\mathbb{Z})$ by the function $F\colon X \longrightarrow \mathrm{H}_1(\mathbb{Z})$ which is defined component-wise by $F(x) = [f(x), g(x), h(x)]$ (for the notation see Example 2.4). The real valued functions f, g, and h are following piecewise constant functions, in which $\beta \in (0, 1/4)$ is defined by $\beta = \alpha \bmod 1/4$:

$$
\begin{aligned}
f(x) &= \chi_{[0,\frac14)\cup[\frac12,\frac34)} - \chi_{[\frac14,\frac12)\cup[\frac34,1)}, \\
g(x) &= \chi_{[0,\frac14)} - \chi_{[\frac34,1)}, \\
i(x) &= \chi_{[\frac{\beta}{2},\frac18)} - \chi_{[\frac18+\frac{\beta}{2},\frac14)}, \\
h(x) &= i(x) + i(x-1/4) + i(x-1/2) + i(x-3/4).
\end{aligned}
\tag{3.1}
$$

The functions f, g, and h fulfil the prerequisites in Example 2.4 and hence $\mathbf{1} \in E(F)$, but the aim for the rest of this section is to show that $E(F) = \mathrm{H}_1(\mathbb{Z})$.

THEOREM 3.3: *Assume that $\alpha \in (0,1)$ is an irrational number with bounded continued fraction coefficients and let F be defined as above. Then the cocycle $F(n,x)$ has essential values $E(F) = \mathrm{H}_1(\mathbb{Z})$ and thus the skew product extension $\mathbf{T}_F$ is ergodic. Any function cohomologous to F also has full essential range, and the set of all functions cohomologous to F is dense in $\mathcal{B}(X, \mathrm{H}_1(\mathbb{Z}))$ with respect to the topology of convergence in measure. Furthermore, functions with* *full essential range are generic, i.e. they form a dense G_δ-set in the complete separable metric space $\mathcal{B}(X, \mathrm{H}_1(\mathbb{Z}))$.*

The proof of Theorem 3.3 relies on the following two lemmas, which have the same hypotheses as the theorem:

LEMMA 3.4: *The cocycle $(f,g)(n,x)$ defined by the function $(f,g)\colon X \longrightarrow \mathbb{Z}^2$ has essential values $E((f,g)) = \mathbb{Z}^2$.*

LEMMA 3.5: *The centre $Z(\mathrm{H}_1(\mathbb{Z})) = \{[0,0,c]\colon c \in \mathbb{Z}\}$ is a subgroup of $E(F)$.*

Proof of Theorem 3.3 given Lemmas 3.4 and 3.5: We fix $[a,b,c] \in \mathrm{H}_1(\mathbb{Z})$ and choose $B \in \mathcal{S}$ with $\mu(B) > 0$ arbitrarily. Lemma 3.4 shows that for some $m \neq 0$ the set $C = B \cap T^{-m}B \cap \{x\colon (f,g)(m,x) = (a,b)\}$ has positive measure. From equation (2.20) and the countability of $\mathrm{H}_1(\mathbb{Z})$ it follows with a suitable $d \in \mathbb{Z}$ that $F(m,x) = [a,b,d]$ for all $x \in D \subset C$ with $\mu(D) > 0$. But Lemma 3.5 implies that the set $E = T^m D \cap T^{m-n} D \cap \{x\colon F(n,x) = [0,0,c-d]\}$ has positive measure and we obtain that $T^{-m}E \subset B \cap T^{-m-n} B \cap \{x\colon F(m+n,x) = [a,b,c]\}$. Hence $[a,b,c] \in E(F)$ and as $[a,b,c]$ was arbitrary it follows that $E(F) = \mathrm{H}_1(\mathbb{Z})$.

The set of integer valued coboundaries is dense in $\mathcal{B}(X,\mathbb{Z})$ with respect to the topology of convergence in measure (cf. [6], Corollary 2), and applied to the three components in $\mathrm{H}_1(\mathbb{Z})$ it follows that the set of coboundaries is dense in $\mathcal{B}(X, \mathrm{H}_1(\mathbb{Z}))$.

The genericity of functions with full essential range follows from the results in [3] for any amenable locally compact second countable group. ∎

Proof of Lemma 3.4: For every $N > 0$ we consider the set of discontinuities of the function $(f,g)(N,\cdot)$, which is equal to

$$\{x \in [0,1)\colon x = -k\alpha \pmod{1/4} \text{ for some } 0 \le k < N\}.$$

We arrange these discontinuities together with the point 1 in increasing order:

$$\gamma_0^{(N)} = 0, \gamma_1^{(N)}, \ldots, \gamma_N^{(N)} = 1/4, \ldots, \gamma_{2N}^{(N)} = 1/2, \ldots, \gamma_{3N}^{(N)} = 3/4, \ldots, \gamma_{4N}^{(N)} = 1.$$

In each of the four intervals $[0,1/4), \ldots, [3/4,1)$ the discontinuities are distributed in the same manner as

$$\{x \in [0,1)\colon x = -k4\alpha \pmod{1} \text{ for some } 0 \le k < N\}, \tag{3.2}$$

but with scaling factor $1/4$. Due to Theorem 23 in [4] an irrational number α has bounded continued fraction coefficients if and only if there exists a constant $c > 0$ so that

$$|\alpha - p/q| < c/q^2$$

has no solution in integers p and q, and so it is obvious that also 4α has bounded continued fraction coefficients. Let $\{q_i\}_{i\geq 0}$ be the strictly increasing sequence of denominators in the convergents $\{p_i/q_i\}_{i\geq 0}$ of 4α; then

$$d/q_i^2 \leq |4\alpha - p_i/q_i| < 1/q_i^2 \quad \text{and} \quad q_i\|q_i 4\alpha\| \geq d \tag{3.3}$$

for all $i \geq 0$ and some constant $d \geq c/4 > 0$. We choose the integer i so that $q_i < N \leq q_{i+1}$ and turn to the distribution of the points in (3.2). For every $0 < l < q_{i+1}$ the inequality $\|l4\alpha\| \geq \|q_i 4\alpha\|$ is obeyed and with equation (2.21) the distance between any two points ξ and η next to each other in the set (3.2) is bounded by

$$d/q_i \leq \|\xi - \eta\| \leq 2/q_i.$$

Thus we have

$$\min_{0\leq k\leq 4N-1}(\gamma_{k+1}^{(N)} - \gamma_k^{(N)}) \geq d/4q_i > d/4N \tag{3.4}$$

and

$$d/2 \leq (\gamma_{k+1}^{(N)} - \gamma_k^{(N)})/(\gamma_{l+1}^{(N)} - \gamma_l^{(N)}) \leq 2/d \tag{3.5}$$

for all $0 \leq k, l < 4N$. Among the $4N$ intervals $[\gamma_k^{(N)}, \gamma_{k+1}^{(N)})$ there must be one with length at most $1/4N$ and therefore

$$\max_{0\leq k\leq 4N-1}(\gamma_{k+1}^{(N)} - \gamma_k^{(N)}) \leq 1/2dN. \tag{3.6}$$

Now we turn to the discontinuities of $(f, g)(N, \cdot)$ arising from the discontinuity of (f, g) at zero, which form the set

$$\{x \in [0, 1) \colon x = -k\alpha \ (\mathrm{mod}\, 1) \text{ for some } 0 \leq k < N\},$$

and denote by $\delta_0^{(N)} = 0, \delta_1^{(N)}, \ldots, \delta_N^{(N)} = 1$ these discontinuities together with 1 in increasing order. By the same method as above we obtain for all $0 \leq k, l < N$ that

$$c/2 \leq (\delta_{k+1}^{(N)} - \delta_k^{(N)})/(\delta_{l+1}^{(N)} - \delta_l^{(N)}) \leq 2/c \quad \text{and} \quad \max_{0\leq k\leq N-1}(\delta_{k+1}^{(N)} - \delta_k^{(N)}) \leq 2/cN.$$

By (3.4), there can be at most $\lfloor 8/cd \rfloor - 1$ discontinuities $\gamma_k^{(N)}, \ldots, \gamma_{k+p}^{(N)}$ between $\delta_l^{(N)}$ and $\delta_{l+1}^{(N)}$ for any $0 \leq l < N$, and the same argument also applies to the discontinuities of $(f,g)(N,\cdot)$ which stem from any other fixed discontinuity of (f,g).

We choose for every $x \in X$ an integer $0 \leq k < 4N$ with $x \in [\gamma_k^{(N)}, \gamma_{k+1}^{(N)}) \colon = I_0^N(x)$ and define for every integer l an interval $I_l^N(x) = [\gamma_m^{(N)}, \gamma_{m+1}^{(N)})$ by $m = k + l \pmod{4N}$. Another set of intervals $J_l^N(x)$ with $x \in J_0^N(x)$ is defined by $J_l^N(x) = T^N(I_l^N(T^{-N}x))$, and the length of all these intervals tends to zero as N goes to infinity.

Assume that B is a Borel set of positive measure and let $M > 0$ be an integer which is set at $M = 1$ for now. By Lebesgue's density theorem there exists an integer $N_0 > 0$ and a Borel set $B' \subset B$ of positive measure with

$$\frac{\mu(B \cap \bigcup_{l=-M}^M I_l^N(x))}{\mu(\bigcup_{l=-M}^M I_l^N(x))} > 1 - \frac{d/4}{2M+1} \quad \text{and} \quad \frac{\mu(B \cap \bigcup_{l=-M}^M J_l^N(x))}{\mu(\bigcup_{l=-M}^M J_l^N(x))} > 1 - \frac{d/4}{2M+1}$$

for all $x \in B'$ and $N \geq N_0$. Now we use that (f,g) is recurrent because F is recurrent, and hence for some $n \geq N_0$ the set

$$C = B' \cap T^{-n}B' \cap \{x \colon (f,g)(n,x) = (0,0)\} \tag{3.7}$$

has positive measure. For every $x \in C \subset B'$ and every integer m with $|m| \leq M$ it results from (3.5) that $\mu(\bigcup_{l=-M}^M I_l^N(x)) \leq (2M+1)(2/d)\mu(I_m^N(x))$ and hence

$$\mu(I_m^n(x) \setminus B) \leq \mu\Big(\bigcup_{l=-M}^{M} I_l^n(x) \setminus B\Big) < \frac{d/4}{2M+1}\mu\Big(\bigcup_{l=-M}^{M} I_l^n(x)\Big) \leq \frac{1}{2}\mu(I_m^n(x)),$$

and the same argument also applies to the intervals $J_l^n(T^n x)$, because $T^n x \in B'$. It follows for all $x \in C$ and $|m| \leq M$ that

$$\frac{\mu(B \cap I_m^n(x))}{\mu(I_m^n(x))} > \frac{1}{2} \quad \text{and} \quad \frac{\mu(B \cap J_m^n(T^n x))}{\mu(J_m^n(T^n x))} = \frac{\mu(T^{-n}B \cap I_m^n(x))}{\mu(I_m^n(x))} > \frac{1}{2},$$

and therefore

$$\mu(B \cap T^{-n}B \cap I_m^n(x)) > 0. \tag{3.8}$$

Our next aim is to show that $(2,1)$ as well as $(2,-1)$ are essential values of (f,g), and this will turn out to be sufficient for $E((f,g)) = \mathbb{Z}^2$. We fix $y \in C$ and observe that $(f,g)(n,x) = (0,0)$ for all $x \in I_0^n(y)$ and $(f,g)(n,x) = (\pm 2, \pm 1)$ for all $x \in I_1^n(y) \cup I_{-1}^n(y)$, because at any discontinuity the functions

f and g have jump heights of ± 2 and ± 1, respectively. If $(f,g)(n, I_1^n(y))$ and $(f,g)(n, I_{-1}^n(y))$ are linearly independent, then inequality (3.8) implies that for some $n' = \pm n$ and $n'' = \pm n$ the sets $B \cap T^{-n'} B \cap \{x\colon (f,g)(n',x) = (2,1)\}$ and $B \cap T^{-n''} B \cap \{x\colon (f,g)(n'',x) = (2,-1)\}$ both have positive measure and we are finished. Otherwise, for appropriately chosen $s = \pm 1$ and $n_0 = \pm n$ the set $\bar{B} = B \cap T^{-n_0} B \cap \{x\colon (f,g)(n_0,x) = (2,s)\}$ has positive measure and we put $B_1 = T^{n_0}\bar{B} \subset B$. Then $B = B_0$ is replaced by B_1 and the construction is iterated with fixed s up to $\lfloor 4/cd \rfloor$ times to obtain $(B_i, n_i), 0 \le i \le \lfloor 4/cd \rfloor$ with

$$B_{i+1} = T^{n_i}(B_i \cap T^{-n_i} B_i \cap \{x\colon (f,g)(n_i,x) = (2,s)\}) \subset B_i \subset B,$$

but we are prematurely successful if some iteration step gives for an integer n' that $\mu(B_i \cap T^{-n'} B_i \cap \{x\colon (f,g)(n',x) = (2,-s)\}) > 0$. If the $\lfloor 4/cd \rfloor$-th iteration step was still not successful, then $M = 1$ is replaced by $M = \lfloor 4/cd \rfloor + 1$ for another iteration. The definitions of f and g show that their jump steps have **equal** sign at 0 and 1/2 and **opposite** sign at 1/2 and 3/4. A discontinuity with equal sign causes a step of $\pm(2,1)$ while a discontinuity with opposite sign causes a step of $\pm(2,-1)$. Starting from a fixed $y_{M-1} \in C_{M-1}$ (the set out of equation (3.7) in the $\lfloor 4/cd \rfloor + 1 = M$-th iteration step) in a suitable direction we have to cross at most $\lfloor 4/cd \rfloor$ discontinuities of $(f,g)(n_{M-1}, \cdot)$ with step $\pm(2,s)$ until crossing a discontinuity with step $\pm(2,-s)$. Adding up all these steps, some of which might be annihilated mutual, shows that $(f,g)(n, I_k^n(x)) = l(2,s) \pm (2,-s)$ with $|k| \le M$ and $|l| < |k|$. If $l = 0$ we are finished due to inequality (3.8), and otherwise we put $n_{M-1} = -\operatorname{sgn}(l) \cdot n$ and achieve that

$$\bar{B}_M = B_{M-1} \cap T^{-n_{M-1}} B_{M-1} \cap \{x\colon (f,g)(n_{M-1},x) = -|l|(2,s) \pm (2,-s)\})$$

has positive measure. With the integer $\bar{n} = \sum_{i=M-|l|-1}^{M-2} n_i$ it results by iteration that $(f,g)(\bar{n}, T^{-\bar{n}}\bar{B}_M) = |l|(2,s)$ and $T^{-\bar{n}}\bar{B}_M \subset B \cap T^{-n_{M-1}-\bar{n}} B$. Hence it follows that

$$(f,g)(n_{M-1} + \bar{n}, T^{-\bar{n}}\bar{B}_M) = |l|(2,s) - |l|(2,s) \pm (2,-s) = \pm(2,-s)$$

and the proof of $\{(2,1),(2,-1)\} \subset E((f,g))$ is finished.

If H denotes the subgroup of $\mathbb{Z}^2$ generated by $(2,1)$ and $(2,-1)$, then the quotient $\mathbb{Z}^2/H$ consists of the four elements $\{(i,0) + H\colon 0 \le i \le 3\}$ and hence is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. From the definition of f and g it is easy to see that $(f,g) + H \equiv (3,0) + H$, a generator of $\mathbb{Z}^2/H$. Applying that T^4 is also an ergodic irrational rotation shows that for every $B \in \mathcal{S}$ with $\mu(B) > 0$ and every $0 \le k < 4$ there is an integer n with

$$\mu(B \cap T^{-4n-k} B \cap \{x\colon ((f,g) + H)(4n+k,x) = k(3,0) + H\}) > 0,$$

and hence $(f,g)+H$ has essential values $\mathbb{Z}^2/H$. By Proposition 3.1 it follows that

$$\mathbb{Z}^2/H = E((f,g)+H) = E((f,g))/H$$

and therefore $E((f,g))+H = E((f,g)) = \mathbb{Z}^2$. ∎

Proof of Lemma 3.5: Now we consider the discontinuities of the function $F(N,\cdot)$ for every integer $N>0$, and from equation (2.20) it is obvious that there is a discontinuity in the first and the second component of $F(N,\cdot)$ at all points $\gamma_i^{(N)}, 0 \le i \le 4N$. Further discontinuities appear with jumps **only** in the third component $h(N,x)+\sum_{r=0}^{N-2} g(T^r x) f(N-r-1, T^{r+1}x)$, and we will show that **at least** one such discontinuity occurs between any two discontinuities of the first and second coordinate. This behaviour, which will be sufficient for $Z(\mathrm{H}_1(\mathbb{Z})) \subset E(F)$, is due to the choice of the function h with jumps at $\beta/2$, $1/8$, $1/8+\beta/2$ and periodic with period $1/4$. Again we arrange these $16N$ discontinuities together with the point 1 in increasing order:

$$\rho_0^{(N)} = 0, \rho_1^{(N)}, \ldots, \rho_{4N}^{(N)} = 1/4, \ldots, \rho_{8N}^{(N)} = 1/2, \ldots, \rho_{12N}^{(N)} = 3/4, \ldots, \rho_{16N}^{(N)} = 1,$$

and observe that in each of the four intervals $[0,1/4), \ldots, [3/4,1)$ these discontinuities are distributed in the same way as

$$\{-k4\alpha, -k4\alpha+2\beta, -k4\alpha+1/2, -k4\alpha+1/2+2\beta \pmod 1 \colon 0 \le k < N\}, \tag{3.9}$$

but with scaling factor $1/4$. If we consider two points

$$\xi = -k4\alpha \pmod 1 \quad \text{and} \quad \eta = -l4\alpha \pmod 1$$

next to each other in the set (3.2) with $0 \le k < l < N$, then there is always a point out of the set in (3.9) between them. Indeed, if $l-k$ is even then

$$\xi - (1/2)(l-k)4\alpha \pmod 1$$

is exactly in distance $1/2$ (i.e. diametrical on the circle) to the point $(\xi+\eta)/2$. Otherwise, if $l-k$ is odd then

$$\xi - (1/2)(l-k)4\alpha = \xi + 2\alpha - (1/2)(l-k+1)4\alpha \pmod 1$$

is either equal to or in distance $1/2$ to $(\xi+\eta)/2$. But the definition of β implies that $2\alpha = 2\beta \pmod{1/2}$ and hence there is always one discontinuity **only** of the third component of $F(N,\cdot)$ exactly at $(\gamma_i^{(N)} + \gamma_{i+1}^{(N)})/2$ for all $0 \le i < 4N$.

In the next step we need to show that the points in the set (3.9) are **sufficiently separated** from each other. Assume that

$$\sigma = -k4\alpha + p/2 + 2q\beta \pmod 1 \quad \text{and} \quad \tau = -l4\alpha + r/2 + 2s\beta \pmod 1$$

with some $0 \leq k < l < N$ and $p, q, r, s \in \{0, 1\}$ are two points next to each other in the set (3.9). With suitably chosen $0 < m < 2N$ it follows that

$$2\|\sigma - \tau\| \geq \|2(\sigma - \tau)\| = \|2(l-k)4\alpha + 4(q-s)\beta \pmod 1\| = \|m4\alpha\|,$$

and the second inequality (3.3) gives that $\|\sigma - \tau\| > d/4N$ and hence

$$\min_{0 \leq k \leq 16N-1} (\rho_{k+1}^{(N)} - \rho_k^{(N)}) > d/16N. \tag{3.10}$$

For every $x \in X$ the intervals $I_l^N(x)$ and $J_l^N(x)$ are defined as in the proof of Lemma 3.4. Then we choose $0 \leq k < 16N$ with $x \in [\rho_k^{(N)}, \rho_{k+1}^{(N)})\colon = K_0^N(x)$ and define for $|l| \leq 1$ an interval $K_l^N(x) = [\rho_m^{(N)}, \rho_{m+1}^{(N)})$ by $m = k + l \pmod{16N}$ and put $L_l^N(x) = T^N(K_l^N(T^{-N}x))$. The considerations above show for all $x \in X$ and $N > 0$ that at least one of the inclusions $K_1^N(x) \subset I_0^N(x)$ and $K_{-1}^N(x) \subset I_0^N(x)$ holds, while $K_l^N(x) \subset \bigcup_{|i| \leq 1} I_i^N(x)$ and $L_l^N(x) \subset \bigcup_{|i| \leq 1} J_i^N(x)$ are trivial for $|l| \leq 1$. From the inequalities (3.6) and (3.10) it results for all $|i|, |l| \leq 1$ that

$$\mu(K_i^N(x))/\mu(I_l^N(x)) > d^2/8 \quad \text{and} \quad \mu(L_i^N(x))/\mu(J_l^N(x)) > d^2/8. \tag{3.11}$$

If B is an arbitrary Borel set of positive measure, then by the density theorem there is an integer $N_0 > 0$ and a Borel set B' of positive measure with

$$\frac{\mu(B \cap \bigcup_{|i| \leq 1} I_i^N(x))}{\mu(\bigcup_{|i| \leq 1} I_i^N(x))} > 1 - \frac{d^2}{48} \quad \text{and} \quad \frac{\mu(B \cap \bigcup_{|i| \leq 1} J_i^N(x))}{\mu(\bigcup_{|i| \leq 1} J_i^N(x))} > 1 - \frac{d^2}{48} \tag{3.12}$$

for all $N \geq N_0$ and $x \in B'$. Now we apply that $F(n, \cdot)$ is recurrent and thus for some $n \geq N_0$ the set $C = B' \cap T^{-n}B' \cap \{x\colon F(n,x) = [0,0,0]\}$ has positive measure. For every $x \in C \subset B'$ and $|l| \leq 1$ the inequalities (3.11) and (3.12) show that

$$\mu(K_l^n(x) \backslash B) \leq \mu\Big(\bigcup_{|i| \leq 1} I_i^N(x) \backslash B\Big) < (d^2/48)\mu\Big(\bigcup_{|i| \leq 1} I_i^N(x)\Big) < (1/2)\mu(K_l^n(x)),$$

and the same holds for $L_l^n(T^n x)$ and $J_i^N(T^n x)$ replacing $K_l^n(x)$ and $I_i^n(x)$ because also $T^n x \in B'$. We conclude for all $x \in C$ and $|l| \leq 1$ that

$$\frac{\mu(B \cap K_l^n(x))}{\mu(K_l^n(x))} > \frac{1}{2} \quad \text{and} \quad \frac{\mu(B \cap L_l^n(T^n x))}{\mu(L_l^n(T^n x))} = \frac{\mu(T^{-n}B \cap K_l^n(x))}{\mu(K_l^n(x))} > \frac{1}{2}$$

and hence

$$\mu(B \cap T^{-n}B \cap K_l^n(x)) > 0. \tag{3.13}$$

If we fix $y \in C$ and choose $l = \pm 1$ with $K_l^n(x) \subset I_0^n(y)$, then there is exactly one discontinuity of $h(n, \cdot)$ between $y \in C$ and any $x \in K_l^n(y)$, while the first and the second component as well as the rest of the third component remain constant. All discontinuities of the function h have a jump height of ± 1 and thus $F(n, K_l^n(y)) = [0, 0, \pm 1]$, and together with inequality (3.13) it follows that $\mu(B \cap T^{-n}B \cap \{x\colon F(n,x) = [0,0,1]\}) > 0$. Now Proposition 3.1 implies that $Z(\mathrm{H}_1(\mathbb{Z})) = \{[0,0,c]\colon c \in \mathbb{Z}\} \subset E(F)$. ∎

Remark 3.6: The paper [8] treats ergodic skew product extensions of irrational rotations by $\mathbb{R}$ and $\mathbb{Z}$, also defined by piecewise constant functions, but without the premise of bounded continued fraction coefficients. Then the separation between the discontinuities of the cocycle is not good enough for the method used in this paper, because a rotation number with unbounded continued fraction coefficients can be approximated well by rationals. In this case the proofs essentially rely on the fact that the integer-valued cocycle $f(n, \cdot)$ assumes just a bounded number of values whenever n is a denominator of a well approximating rational. This follows from the Denjoy–Koksma inequality, but such an inequality does not hold for the third component in the discrete Heisenberg group because of its faster growth.

Example 3.7 (The essential range is not a cohomology invariant in $\mathrm{H}_1(\mathbb{Z})$*; cf. [1] for examples in* $\mathrm{GL}_2(\mathbb{R})$ *and* $\mathrm{SO}(3)$*):* We define $H(x) = [f(x), 2f(x), 2h(x)]$ by the functions f and h out of (3.1) and put $B(x) = [0, \chi_{[0,\frac{1}{2})}, 0]$. It follows by induction that $H(n,x) \in \{[k, 2k, 2l]\colon k, l \in \mathbb{Z}\}$ for all $n \in \mathbb{Z}$ and $x \in X$. The same methods as in the proofs of Lemmas 3.4 and 3.5 show that $E(H) = \{[k, 2k, 2l]\colon k, l \in \mathbb{Z}\}$, while the cohomologous cocycle $K(n,x) = B(T^n x) \cdot H(n,x) \cdot B(x)^{-1}$ defined by the function $K(x) = B(Tx) \cdot H(x) \cdot B(x)^{-1}$ has no essential values. Indeed, if $C \subset [0, 1/2)$ and $m \in \mathbb{Z}$ then $K(m,x) \in \{[k, 2k, 2l+1]\colon k, l \in \mathbb{Z}\}$ for all $x \in C \cap T^{-m}C$, but if $D \subset [1/2, 1)$ and $n \in \mathbb{Z}$ then $K(n,x) \in \{[k, 2k, 2l]\colon k, l \in \mathbb{Z}\}$ for all $x \in D \cap T^{-n}D$.

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